Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$

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Bounding Hull

- Should enclose the entire set
- Eg: Convex hull, Non-Convex hull and positive $\alpha$-hull

(a) Convex hull  
(b) Non-Convex hull  
(c) Positive $\alpha$-hull

Concave hull of a set of freeform closed surfaces in $R^3$
Convex Hull

**Definition**

*The convex hull of a set S in the smallest area convex enclosure enclosing the set*

**Figure**: Convex Hull of a set of points
Bounding Hull

Non-Convex Hull

- Convexity constraint relaxed
- Gives a tighter enclosure to the set

*Figure*: Non-Convex Hull

Concave hull of a set of freeform closed surfaces in $R^3$
Contributions

- The definition for concave hull for a set of surfaces in $R^3$.
- Simply connected surfaces having genus=0, genus$>0$ are considered.
- Concave hull is explained by means of a rubber band analogy for intersecting and disjoint surfaces.
Concave Hull

Definition

Concave hull of a set of surfaces is the enclosing concave surface with smallest volume.

Figure: Set of surfaces

Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$
Combinations

- One surface lying completely outside the other
- Surfaces that intersect each other.
- One surface lying completely inside the other.
Rubber sheet stretched around two surfaces

Figure: Surfaces Outside each other
Antipodality Constraint

Lemma

The distance between two closed $C^1$ non intersecting surfaces is minimum only when the normals of the corresponding points are opposite to each other (i.e., antipodal.)

\[
\begin{align*}
\left\langle \frac{\partial S_1(u_1, v_1)}{\partial u_1}, S_1(u_1, v_1) - \frac{S_1(u_1, v_1 + S_2(u_2, v_2))}{2} \right\rangle &= 0, \\
\left\langle \frac{\partial S_1(u_1, v_1)}{\partial v_1}, S_1(u_1, v_1) - \frac{S_1(u_1, v_1 + S_2(u_2, v_2))}{2} \right\rangle &= 0, \\
\left\langle \frac{\partial S_2(u_2, v_2)}{\partial u_2}, S_2(u_2, v_2) - \frac{S_1(u_1, v_1 + S_2(u_2, v_2))}{2} \right\rangle &= 0, \\
\left\langle \frac{\partial S_2(u_2, v_2)}{\partial v_2}, S_2(u_2, v_2) - \frac{S_1(u_1, v_1 + S_2(u_2, v_2))}{2} \right\rangle &= 0,
\end{align*}
\]
Antipodality Constraint

- Antipodal lines between surfaces found using constraint equations
- Minimum antipodal line is selected

(a) A pair of surfaces outside each other
(b) Output showing all antipodal lines
(c) Output showing MAL

Concave hull of a set of freeform closed surfaces in $R^3$
Rubber sheet analogy for intersecting Surfaces

(a) A pair of surfaces that intersect each other
(b) Convex Hull of the surfaces
(c) Pushing the convex hull of the curves

Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$
Boolean union

- Intersection curve between surfaces acts as trimming curves
- Portions of one surfaces inside another is removed

Figure: (a) Two intersecting surfaces $S_1$ and $S_2$. (b) Intersection curve as trimmed curve. (c) Minimum Area Hull.
Types of intersecting surfaces

- Simply connected object with genus $= 0$
- Simply connected object with genus $> 0$
- Multiply connected object
Genus=0

(a) Two surfaces of genus 0
(b) MVE formed by boolean union
Genus $> 0$ surfaces

- Rubber Sheet analogy will not allow piercing of the genus
- To have uniqueness, the genus is filled with a degenerate ruled surface patch

(c) Pair of dumbell-like surfaces.
(d) Zero volume degenerate ruled surface.

Concave hull of a set of freeform closed surfaces in $R^3$
Degenerate ruled surface patch

- Find intersection curves

Figure: Intersecting surfaces curves of two dumbell shapes

Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$
Degenerate ruled surface patch

- Find intersection curves
- Perform boolean union

Figure: Boolean union of two dumbbell shapes
Degenerate ruled surface patch

- Find intersection curves
- Perform boolean union
- Find the minimum geodesic curve between the two intersecting curves on either surface

**Figure**: Geodesic curve connecting the intersection curves
Degenerate ruled surface patch

- Find intersection curves
- Perform boolean union
- Find the minimum geodesic curve between the two intersecting curves on either surface
- Fit a ruled surface patch having the geodesic curves as boundary curves

Figure: Zero volume Patch

Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$
Attributes of DRSP

- Obtained by valid topological operation on circular disc
- In $\mathbb{R}^3$, the DRSP is equivalent to topological discs ($\mathbb{R}^2$)
- For higher dimensions $\mathbb{R}^n$, DRSP will be lower dimensional entity
Multiply connected surfaces

- Ray tracing approach used to remove interior region

Figure: Input surfaces multiply connected
Multiply connected surfaces

- Ray tracing approach used to remove interior region

(a) Two simply-connected surfaces intersecting.
(b) Boolean union resulting in multiply-connected surfaces.
(c) Output concave hull.

Concave hull of a set of freeform closed surfaces in $R^3$
Algorithm for a set of surfaces I

Determine surfaces that intersect each other.
Find sets of surfaces that intersect. Let $B_i$ denote each set, and each $B_i$ will consist of surfaces from $C$.

for each set $B_i$ of intersecting curves do
   Perform Boolean union.
   Represent each Boolean union as a single surface (say $BU_i$).
   if $\{BU_i\}$ is simply connected with genus $> 0$ then
      Fit a degenerate ruled surface path
   else
      Eliminate regions.
   end if
end for
Algorithm for a set of surfaces II

end for
Let \( C' = \{\{BU_i\} \cup \{C - \{B_i\}\}\}\).
for each surface \( \in C' \) do
    Compute MAP to all other surfaces in the set.
end for
if All surfaces in \( C' \) are outside each other then
    Use surfaces as nodes and MAPs’ as distances, find the minimum spanning tree (MST).
    Return MST as concave hull.
else
    Return \( C' \) as Minimum Volume Enclosure.
end if
Algorithm

Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$
Algorithm

Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$
Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$
Algorithm

Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$
Limitations

- DRSP intersecting another surface

Figure: Surface intersecting DRSP.
Limitations

- DRSP between set of convex surfaces

Trimmed surface patch has to be employed to construct DRSP
Limitations

- DRSP cannot be constructed for non-monotone curves
Results

(a) Test surfaces

(b) MVE of the surfaces

Concave hull of a set of freeform closed surfaces in $\mathbb{R}^3$
Algorithm to compute concave hull for a set of surfaces in $\mathbb{R}^3$

Simply connected surfaces of genus $= 0$, genus $> 0$ have been considered

Surfaces are not sampled for computation purpose

All the implementation in this work has been carried out using IRIT solid modeling kernel

The constraint equations are carried out using multi-variate solver in IRIT

Limitations of the current technique is also mentioned
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