Trajectory-tracking control of a planar 3-RR parallel manipulator

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Abstract
Parallel manipulators have higher payload capacity, higher mechanical rigidity and better accuracy than their serial counterparts. In order to utilize all these advantages, an effective control scheme is required. But the highly non-linear dynamic model and the presence of passive links in parallel manipulators make it difficult to develop a control scheme which can handle these non-linearities and can be generally applied to any parallel manipulator. This paper presents the trajectory-tracking control scheme of a planar 3-RR parallel manipulator for pre-determined singularity-free-paths. The control scheme presented in this paper is general and can be applied to any parallel manipulator.

1 Introduction
The 3-RR is a planar parallel manipulator with three-degrees-of-freedom. It can be used for many applications, for example, pick and place, profile-cutting etc.

The way the control scheme is developed in order to handle the highly non-linear terms and coupling in the dynamic model of a parallel manipulator is very important. Dynamic model-based control scheme employing the computed torque control algorithm is successfully being used (see for e.g. [?], [?], [?]) in trajectory-tracking for parallel manipulators. We have developed a similar control scheme on the 3-RRR manipulator such that it can be applied to serial, parallel and hybrid manipulators.

In the following sections we present the kinematics, dynamics and controls of the manipulator. We reduce the dynamic model in the full configuration
space to the actuator space, with which we can treat any parallel manipulator equivalent to a serial manipulator. With this dynamic model, we command the manipulator to follow a particular path. We add disturbances to the model simulating parametric uncertainties to make it more realistic. The same simulation we performed at higher speed with no disturbances added. Numerical results illustrate the formulations presented.

2 Kinematic model of the 3-RRR

In this section, we present the zeroth and first order kinematics of the manipulator.

2.1 Geometry of the manipulator

The 3-RRR planar parallel manipulator is shown in figure (1).

![Figure 1: Schematic of the 3-RRR planar parallel manipulator](image)

The “end-effector” of the manipulator is an equilateral triangle, $p_1 p_2 p_3$, with each side of length $a$, which is connected to a fixed base $b_1 b_2 b_3$, which also forms an equilateral triangle with $b$ as the length of each side. The vertices $p_i$ are connected to the corresponding base points of manipulator $b_i$ by three identical fingers. Each finger comprises of two links, of length $l_i$ and $r_i$ respectively, where $i = 1, 2, 3$. All joints are revolute joints. The actuators are located at $b_i$, and therefore the links $l_1, l_2, l_3$ are called the active links and $r_1, r_2, r_3$ the passive links. The platform has three degrees-of-freedom. These are the two translations in the plane parallel to the fixed base and a rotation about the normal to this plane.
2.2 Forward kinematics

There are three active (i.e., actuated) variables represented as $\theta = (\theta_1, \theta_2, \theta_3)^T$ and four passive (i.e., non-actuated) variables represented as $\phi = (\phi_1, \phi_2, \phi_3, \alpha)^T$. The full configuration space can be represented by $q$, where, $q = (\theta^T, \phi^T)$.

Note that the orientation of the moving platform, $\alpha$, is included in the passive variable $\phi$, since like $\phi_1, \phi_2, \phi_3$, it is also determined indirectly through the solution of the forward kinematics of the manipulator.

2.2.1 Loop closure equations

The joint variables are related to the position and orientation of the end-effector by the constraints imposed by the joints. There are three possible closed-loops: (1) starting from point $b_1$ to point $b_2$ via points $p_1, p_2$, (2) starting from point $b_1$ to point $b_3$ via points $p_1, p_3$, (3) starting from point $b_2$ to point $b_3$ via points $p_2, p_3$. The positions for all the points included in the three loops are as follows:

$$b_1 = (0, 0)^T, b_2 = (b, 0)^T, b_3 = \left( \frac{b}{2}, \frac{\sqrt{3}}{2}b \right)^T$$

The position of the point $p_2$ defined in terms of $l_1, r_1, a, b_1$ is given by:

$$p_{2b_1} = (l_1 \cos \theta_1 + r_1 \cos \phi_1, l_1 \sin \theta_1 + r_1 \sin \phi_1)^T$$

The position of the point $p_2$ defined in terms of $l_2, r_2, b_2$ is given by:

$$p_{2b_2} = (r_2 \cos \phi_2 + l_2 \cos \theta_2 + x_{b_2}, r_2 \sin \phi_2 + l_2 \sin \theta_2 + y_{b_2})^T$$

Similarly, the position of the point $p_3$ starting from point $b_2$ through $l_2, r_2$ is given by:

$$p_{3b_2} = (x_{b_2} + l_2 \cos \theta_2 + r_2 \cos \phi_2 + a \cos \left( \frac{2\pi}{3} + \alpha \right), l_2 \sin \theta_2 + r_2 \sin \phi_2 + a \sin \left( \frac{2\pi}{3} + \alpha \right))^T$$

Starting from $b_3$ the position of the point $p_3$ through $l_3, r_3$ is given by:

$$p_{3b_3} = (r_3 \cos \phi_3 + l_3 \cos \theta_3 + x_{b_3}, r_3 \sin \phi_3 + l_3 \sin \theta_3 + y_{b_3})^T$$
Considering the first loop, i.e., $b_1$ to $b_2$ via points $p_1, p_2$, the loop closure can be written as:

$$p^{b_1}_2 - p^{b_2}_2 = 0 \quad (1)$$

Writing the loop-closure equations in terms of $X$ and $Y$ coordinates as we get:

$$\eta_1 = l_1 \cos \theta_1 + r_1 \cos \phi_1 + a \cos \alpha - r_2 \cos \phi_2 - l_2 \cos \theta_2 - x_{b2} = 0 \quad (2)$$

$$\eta_2 = l_1 \sin \theta_1 + r_1 \sin \phi_1 + a \sin \alpha - r_2 \sin \phi_2 - l_2 \sin \theta_2 = 0 \quad (3)$$

Similarly, considering the second loop, i.e., $b_2$ to $b_3$ via points $p_2, p_3$, the loop closure can be written as:

$$p^{b_2}_3 - p^{b_3}_3 = 0 \quad (4)$$

The corresponding loop-closure equations in terms of $X$ and $Y$ coordinates can be written as:

$$\eta_3 = b_{x2} + l_2 \cos \theta_2 + r_2 \cos \phi_2 + a \cos \left(\frac{2\pi}{3} + \alpha\right) - r_3 \cos \phi_3 - l_3 \cos \theta_3 - x_{b3} = 0 \quad (5)$$

$$\eta_4 = l_2 \sin \theta_2 + r_2 \sin \phi_2 + a \sin \left(\frac{2\pi}{3} + \alpha\right) - r_3 \sin \phi_3 - l_3 \sin \theta_3 - y_{b3} = 0 \quad (6)$$

Put together, the loop-closure constraints can be written as:

$$\eta(q) = \eta(\theta, \phi) = 0, \quad (7)$$

where$^1$,

$$\eta = (\eta_1, \eta_2, \eta_3, \eta_4)^T$$

The forward kinematics problem of the manipulator refers to finding the solutions $\phi = \phi(\theta)$ of equation (7).

$^1$We are not considering the last loop, as the constraint equations resulting from the third loop are dependent on the other four independent constraint equations resulting from the rest of the two loops.
2.2.2 Solution of the loop-closure equations

Equations (2, 3) can be solved linearly for \( \sin \phi_1 \) and \( \cos \phi_1 \) to obtain:

\[
\cos \phi_1 = -\frac{l_1 \cos \theta_1 - a \cos \alpha + r_2 \cos \phi_2 + l_2 \cos \theta_2 + b_{x,2}}{r_1}
\]
\[
\sin \phi_1 = -\frac{l_1 \sin \theta_1 - a \sin \alpha + r_2 \sin \phi_2 + l_2 \sin \theta_2}{r_1}
\] (8)

From these, \( \phi_1 \) is obtained uniquely:

\[
\phi_1 = \text{atan2}(\sin \phi_1, \cos \phi_1)
\] (9)

where \( \text{atan2}(\cdot, \cdot) \) represents the two-argument arc-tangent function.

All the three fingers of the manipulator are considered as identical i.e., \( l_1 = l_2 = l_3 = l \) and \( r_1 = r_2 = r_3 = r \). Eliminating \( \phi_1 \) by squaring and adding equations (8), we get an equation \( f_1(\theta, \phi_2, \alpha) = 0 \), which is of the form:

\[
A \cos \psi + B \sin \psi + C = 0,
\] (10)

where,

\[
A = -2br^3 + 2ar^3 \cos \alpha + 2lr^3 \cos \theta_1 - 2lr^3 \cos \theta_2
\]
\[
B = 2ar^3 \sin \alpha + 2lr^3 \sin \theta_1 - 2lr^3 \sin \theta_2
\]
\[
C = 2l^2 r^2 \cos(\theta_1 - \theta_2) + 2alr^2 \cos(\alpha - \theta_2)
- 2alr^2 \cos(\alpha - \theta_1) + 2ab r^2 \cos \alpha + 2br^2 \cos \phi_1
- 2br^2 \sin \phi_2 - (a^2 + b^2 + 2l^2) r^2
\]

Similarly, from equations (5, 6), we obtain \( \phi_3 \) and another equation \( f_2(\theta, \phi_2, \alpha) = 0 \), which is of the same form as equation (10), with \( A, B \) and \( C \) given as:

\[
A = -br^3 + ar^3 \cos \alpha - 2lr^3 \cos \theta_2 + 2lr^3 \cos \theta_3
+ \sqrt{3}ar^3 \sin \alpha
\]
\[
B = \sqrt{3}br^3 - \sqrt{3}ar^3 \cos \alpha + ar^3 \sin \alpha - 2lr^3 \sin \theta_2
+ 2lr^3 \sin \theta_3
\]
\[
C = alr^2 \cos(\alpha - \theta_2) - alr^2 \cos(\alpha - \theta_3)
+ 2l^2 r^2 \cos(\theta_2 - \theta_3) + \sqrt{3}alr^2 \sin(\alpha - \theta_2)
+ \sqrt{3}alr^2 \sin(\alpha + \theta_3) + \sqrt{3}br^2 \sin \theta_2 - \sqrt{3}br^2 \sin \theta_3
+ br^2 \cos \theta_3 + 2ab r^2 \cos \alpha - br^2 \cos \theta_2
- (a^2 + b^2 + 2l^2) r^2
\]
Incidentally, $f_1, f_2$ are linear in $\cos \phi_2, \sin \phi_2$ and therefore applying a similar procedure as above, we solve for $\phi_2$, and obtain the following equation containing only one unknown, $\alpha$:

$$f(\theta, \alpha) = 0$$ (11)

This trigonometric equation is converted into an algebraic equation by substituting the trigonometric terms $\cos \alpha$ and $\sin \alpha$ as follows (see, e.g., [?]):

$$\cos \alpha = \frac{1 - t_{\alpha}^2}{1 + t_{\alpha}^2}, \quad \sin \alpha = \frac{2t_{\alpha}}{1 + t_{\alpha}^2},$$

where,

$$t_{\alpha} = \tan \left(\frac{\alpha}{2}\right)$$

Using the above relations, after rationalization and some symbolic manipulations, we get six degree polynomial in $t_{\alpha}$:

$$a_0t_{\alpha}^6 + a_1t_{\alpha}^5 + a_2t_{\alpha}^4 + a_3t_{\alpha}^3 + a_4t_{\alpha}^2 + a_5t_{\alpha} + a_6 = 0$$ (12)

The coefficients, $a_i$, $i = 0, \ldots, 6$, are obtained as closed-form expressions of the architecture parameters ($a, b, l, r$) and the motors input, $\theta$. For a given set of numerical values of the inputs, equation (12) can be solved numerically for all the six values of $t_{\alpha}$. Therefore all the corresponding values of $\alpha$ and $\phi_i$ can be obtained uniquely, leading to the 6 sets of solutions to the forward kinematics problem.

### 2.3 Velocity kinematics

Referring to figure (1), coordinates of the center point $p$, of the moving platform are function of the all the joint variables and can be expressed as:

$$p = p(q) = p(\theta, \phi)$$ (13)

Differentiating equation (13) with respect to time to get:

$$\dot{p} = \frac{\partial p}{\partial \theta} \dot{\theta} + \frac{\partial p}{\partial \phi} \dot{\phi}$$

Or,

$$\dot{p} = J_{p\theta} \dot{\theta} + J_{p\phi} \dot{\phi}$$ (14)
where,

\[ J_{p\theta} = \frac{\partial p}{\partial \theta}, \quad J_{p\phi} = \frac{\partial p}{\partial \phi} \]

From equation (14), to calculate \( \dot{p} \), we need to find out passive joint rates. In order to do so we differentiate equation (7) with respect to time to get:

\[ \frac{\partial \eta}{\partial \theta} \dot{\theta} + \frac{\partial \eta}{\partial \phi} \dot{\phi} = 0 \]

\[ \Rightarrow \dot{\phi} = -\frac{\partial \eta^{-1}}{\partial \phi} \frac{\partial \eta}{\partial \theta} \dot{\theta}, \quad \text{det} \left( \frac{\partial \eta}{\partial \phi} \right) \neq 0 \]

\[ \Rightarrow \dot{\phi} = -J_{\eta\phi}^{-1} J_{\eta\theta} \dot{\theta} \]

where,

\[ J_{\eta\theta} = \frac{\partial \eta}{\partial \theta}, \quad J_{\eta\phi} = \frac{\partial \eta}{\partial \phi} \]

Therefore, we can write:

\[ \dot{\phi} = J_{\phi\theta} \dot{\theta}, \quad \text{det} \left( J_{\eta\phi} \right) \neq 0 \]

(15)

Substituting \( \dot{\phi} \) in equation (14), we get:

\[ \dot{p} = J_{p\theta} \dot{\theta} + J_{p\phi} J_{\phi\theta} \dot{\theta} \]

\[ \dot{p} = J_v \dot{\theta} \]

where,

\[ J_v = J_{p\theta} + J_{p\phi} J_{\phi\theta} \]

(16)

As we can calculate \( \dot{\phi} \) using equation (15), we can always write:

\[ \dot{p} = \frac{\partial p}{\partial q} \dot{q} = J_{pq} \dot{q}, \quad \text{where} \ J_{pq} = \frac{\partial p}{\partial q} \]

(17)

All the Jacobian matrices appearing above can be computed at any configuration after the forward kinematics problem has been solved. Therefore, the linear velocity of any point of interest, can be obtained by the above process, if the joint velocity vector, \( \dot{\theta} \), is known. With the knowledge of the position and velocities, we now proceed to formulate the dynamic model of the manipulator.
2.4 Singularity condition

Due to its closed-loop structure, the 3-RRR has three types of singularities [?] i.e., (a) inverse or loss-type singularity, (b) forward or gain-type singularity, (c) combined singularity with loss-type and gain-type singularity together.

Among these three types, gain type singularity requires special attention as it restricts the motion of the manipulator within the workspace. The condition for gain type singularity for 3-RRR manipulator is given in a simplified form in [?]:

\[ \det(J_{\eta\phi}) = ar^3 \{ \sin(\phi_1 - \alpha) \sin(\phi_2 - \phi_3) + \sin(\phi_1 - \phi_2) \sin \left( \alpha + \frac{2\pi}{3} - \phi_3 \right) \} = 0, \] 

(18)

where, \( J_{\eta\phi} = \frac{\partial \eta}{\partial \phi} \). We define another singularity measure, \( S(\phi) \) which is independent of the scale of the manipulator:

\[ S(\phi) = \frac{\det(J_{\eta\phi})}{ar^3} \] 

(19)

3 Dynamic model of the 3-RRR

The dynamic model is an essential part of the control scheme as it is used for feedback linearisation of the equation of motion in the control scheme. We use the Lagrangian framework to develop the equation of motion.

3.1 Constrained Lagrangian formulation in the full configuration space

Let \( T_i \) be the kinetic energy of the \( i \)th link. Then the total kinetic energy \( T \) of the manipulator can be written as:

\[ T = \sum_{i=1}^{7} T_i, \quad T_i = \frac{1}{2} v_{p_{c_i}}^T m_i v_{p_{c_i}} + \frac{1}{2} \omega_i^T I_{c_i} \omega_i, \] 

(20)

where, \( m_i \) is the mass, \( v_{p_{c_i}} \) is the velocity of the mass center \( p_{c_i} \), \( \omega_i \) is the angular velocity and \( I_{c_i} \) is the moment of inertia of the \( i \)th link about its center with reference to the body-fixed frame. Using equation (17), we can write

\[ v_{p_{c_i}} = J_{pq_i} \dot{q}_i, \] 

(21)
and similarly, $\omega_i = J_{\omega i} \dot{q}$. Therefore, we rewrite $T_i$ as:

$$T_i = \frac{1}{2} q^T J_{pqi}^T m_i J_{pqi} \dot{q} + \frac{1}{2} q^T J_{\omega i}^T I_{ci} J_{\omega i} \dot{q} \tag{22}$$

From equation (22), the mass matrix of the system can be obtained as:

$$M = \sum_{i=1}^{7} M_i \tag{23}$$

where,

$$M_i = J_{pqi}^T m_i J_{pqi} + J_{\omega i}^T I_{ci} J_{\omega i}$$

Further, the matrix of centripetal and Coriolis’s force, $C$, can be computed from the mass matrix (see, for e.g., [?]):

$$C_{ij} = \frac{1}{2} \sum_{k=1}^{n} \left( \frac{\partial M_{ik}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i} \right) \dot{q}_k \tag{24}$$

With these elements, we can now put together the Lagrangian equation of motion for the manipulator as:

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} = \tau + \tau_c, \tag{25}$$

where, $\tau$ represents the applied torques and $\tau_c$ represents the constraint forces and moments associated with the loop-closure constraints. Note that the potential term $G(q)$ representing the forces due to gravity is not taken into account in equation (25) as the mechanism is considered in horizontal $XY$ plane for all the simulations.

It is well-known that $\tau_c$ can be written as $J_{\eta q}^T \lambda$ (see, e.g., [?]), where:

$$J_{\eta q} = \frac{\partial \eta}{\partial q}$$

Therefore, the equation of motion of the manipulator can be written as:

$$M(q) \ddot{q} + C(q, \dot{q}) \dot{q} = \tau + J_{\eta q}^T \lambda, \tag{26}$$

where, $\lambda$ is a vector of Lagrange multipliers. It is possible to explicitly evaluate $\lambda$ from equation (26) by obtaining $\dot{q}$ from it and substituting it in the second derivative of the constraint equation $\eta(q) = 0$ (see, e.g., [?])

$$\lambda = -(J_{\eta q} M^{-1} J_{\eta q}^T)^{-1} \{J_{\eta q} \dot{q} + J_{\eta q} M^{-1}(\tau - C \dot{q})\},$$

where,

$$\det(J_{\eta q} M^{-1} J_{\eta q}^T) \neq 0.$$
3.2 Reduction of the model to the actuator space

In order to eliminate $\lambda$ from equation (26), first we rewrite it as:

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} - \tau - J_{qq}^T \lambda = 0$$ (27)

Pre-multiplying equation (27) by $\dot{q}^T$, we get:

$$\dot{q}^T M(q)\ddot{q} + \dot{q}^T C(q, \dot{q})\dot{q} - \dot{q}^T \tau - \dot{q}^T J_{qq}^T \lambda = 0$$ (28)

Note that:

$$\frac{d\eta}{dt} = 0 \Rightarrow J_{qq} \dot{\eta} = 0, \quad (29)$$

From equation (29),

$$\dot{q}^T J_{qq}^T = 0$$

Therefore, in equation (28), $\dot{q}^T J_{qq}^T \lambda = 0$ for any finite $\lambda$. This step, therefore, effectively eliminates the constraint force term, and creates an equivalent unconstrained dynamic system as follows:

$$\dot{q}^T M(q)\ddot{q} + \dot{q}^T C(q, \dot{q})\dot{q} - \dot{q}^T \tau = 0,$$ (30)

Using equation (15), we can write $\dot{q}$ as:

$$\dot{q} = \begin{pmatrix} \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} I_3 \\ J_{\theta \phi} \end{pmatrix} \dot{\theta} = J_{q\theta} \dot{\theta}, \quad (31)$$

where, $I_3$ is the $3 \times 3$ identity matrix.

Differentiating equation (31) with respect to time, we get:

$$\ddot{q} = J_{q\theta} \ddot{\theta} + J_{q\phi} \dot{\theta} \quad (32)$$

Substituting $\dot{q}$, $\ddot{q}$ from equation (31, 32), into equation (30), we get:

$$\dot{\theta}^T (J_{q\theta}^T M J_{q\theta} \ddot{\theta} + J_{q\theta}^T (MJ_{q\theta} + CJ_{q\theta}) \dot{\theta} - J_{q\theta}^T \tau) = 0$$ (33)

Equation (33), is satisfied by any arbitrary $\dot{\theta}$ and therefore,

$$J_{q\theta}^T M J_{q\theta} \ddot{\theta} + J_{q\theta}^T (MJ_{q\theta} + CJ_{q\theta}) \dot{\theta} - J_{q\theta}^T \tau = 0$$ (34)
Equation (34) can be rewritten more compactly as:

\[ M \ddot{\theta} + C \dot{\theta} = \tau \theta \]  

(35)

where \( M \theta = J_{q\theta}^T M J_{q\theta} \) is the mass matrix of the manipulator as reflected on the active coordinates alone, \( C \theta = J_{q\theta}^T \left( M J_{q\theta} + C J_{q\theta} \right) \) is the corresponding centripetal and Corioli’s term, and \( \tau \theta = J_{q\theta}^T \tau \) is the vector of joint torques.

Expressed in this way, the dynamic model of the parallel manipulator is no different from that of an equivalent serial manipulator with the same actuators. It also has a compact form, i.e., it has only 3 ordinary differential equations as opposed to 7 in the original system (26). However, there is a very serious distinction, which is apparently buried in the matrix manipulations above. The model (35) is only valid when the mapping \( \dot{q} = J_{q\theta} \dot{\theta} \) is well-defined, i.e., when \( J_{\phi\theta} \) exists, i.e., when \( J_{\eta\phi} \) is non-singular. Physically, this means that the model is valid only when the manipulator is not at a gain-type singularity. Further, integration of this model only updates the active variable, \( \theta \), while the matrices \( M \theta, C \theta \) etc. are dependent on \( \theta \) as well as \( \phi \). Therefore, to complete the update of the model, every integration step has to be followed by a step of forward kinematics, so as to update the passive variables \( \phi \). From an implementation point of view, this also introduces another complexity: while computing the passive variables at any time step, one has to ensure that the branch of solution chosen (among the possible 6) is the same as the original, i.e., at the initial time. This branch-following in forward kinematics is essential in successfully implementing the equivalent model.

4 Formulation of the control scheme

In this section, we follow a model-based control approach for achieving trajectory tracking along a pre-computed non-singular-path. The \( M \theta, C \theta \) matrices in equation of motion of 3-RRR manipulator are highly non-linear. Applying a linear control scheme is not expected to produce accurate results. Therefore, we have implemented a control scheme in which we use the estimate of the plant to cancel out the non-linear terms, through feedback linearisation.
4.1 Model-based trajectory-tracking control of the 3-RRR manipulator

4.1.1 Feedback linearisation

Given a trajectory we can calculate the desired set of active joints $\theta_d$ and their derivatives and $\dot{\theta}_d$ (where subscript ‘d’ represents the desired value) from inverse kinematics.

Referring to figure (2). The desired and the initial value of $\theta$ and $\dot{\theta}$ is compared with the desired one to get error $e(t)$ and $\dot{e}(t)$, where $e(t) = (\theta(t) - \theta_d(t))$. We apply a simple linear control strategy to get the value of torque required to be provided to the motors to compensate this error and hence follow the desired trajectory.

4.1.2 Assumptions

- The model is deterministic with no parametric uncertainties.
- There are no unmodeled dynamics e.g. friction.
- Full-state feedback is available.

We implement the control law partitioning (see for e.g. [?]), by assuming that the output of the controller can be written as:

$$\tau_\theta = \alpha \tau'_\theta + \beta,$$

where,

$$\alpha = M_\theta, \quad \beta = C \dot{\theta},$$

Substituting $\tau_\theta$, $\alpha$, $\beta$ in equation (35), we get a linear equation:

$$\tau'_\theta = \ddot{\theta}$$

Linear control strategy can now be applied to the system. If we choose the $\tau'_\theta$ as:

$$\tau'_\theta = \ddot{\theta}_d(t) - K_pe(t) - K_v\dot{e}(t),$$

the final error dynamic equation can be written as:

$$\ddot{e}(t) + K_pe(t) + K_v\dot{e}(t) = 0,$$

where, $K_p$ represents the proportional gain and $K_v$ represents the derivative or velocity gain.
4.1.3 Linear feedback

We assume $\mathbf{K}_p$ and $\mathbf{K}_v$ as two positive definite diagonal matrices with the specific case of critical damping:

$$\mathbf{K}_p = k_p \mathbf{I}_3, \quad \mathbf{K}_v = k_v \mathbf{I}_3,$$

where, $k_v = 2\sqrt{k_p}$. The values of $k_p$ is chosen to be scalar.

4.2 Feedback loop

![Figure 2: Model-based-dynamic control of 3-RRR manipulator](image)

Referring to figure (2), $\alpha$ that we get by branch following is used to perform the forward kinematics to get the rest of the passive joint angles. We have the full set of configuration variables now with $\theta$ coming directly from the feedback and the passive variables $\phi$ coming from the branch-following and forward kinematics.

5 Results of numerical simulations

The theoretical developments described above illustrated through a series of numerical simulations, as detailed in this section. We present the results of three different cases. The coordinates of the base points of the manipulator are:

$$x_{b1} = y_{b1} = 0, \quad x_{b2} = b, \quad y_{b2} = 0, \quad x_{b3} = b/2, \quad y_{b3} = b/\sqrt{3},$$

where, $b = 0.5^2$. All the links are taken in proportion with the base link length i.e., $l = b/2, r = b/3$ and $a = b/4$. The details of the geometric pa-

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2 All the linear entities are in meters and all the angles are in radians unless specified otherwise.
rameters and inertia of the manipulator are as follows:

<table>
<thead>
<tr>
<th>link</th>
<th>length</th>
<th>mass (Kg)</th>
<th>moment of inertia $\times 10^{-3}$ (Kg-m$^2$)</th>
</tr>
</thead>
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<td>$l$</td>
<td>0.2500</td>
<td>0.4680</td>
<td>2.4380</td>
</tr>
<tr>
<td>$r$</td>
<td>0.1667</td>
<td>0.3120</td>
<td>0.7220</td>
</tr>
<tr>
<td>$a$</td>
<td>0.1250</td>
<td>0.2340</td>
<td>0.3050</td>
</tr>
</tbody>
</table>

The numerical simulations pertain to the tracking of a circle in the $XY$ plane.

5.1 Tracing the non-singular circle

In the first case we trace a non-singular circle of radius $r_c = 0.05$ and center at $(x_c, y_c)^T = (0.25, 0.21)^T$ with a average speed $v = 0.2$ m/sec. The desired orientation of the platform $\alpha_d(t)$ is set to zero. The values of control gains, $K_p$ and $K_v$, are selected through numerical simulations and for this particular path $k_p$ was taken as 1000. Initial configuration of the manipulator in terms of $x, y, \alpha$ is 0.20, 0.21, 0 respectively. The translational motion along the trajectory is described by the third-order polynomial with respect to time.

Figure (3) shows that the circle traced is non-singular as it is not cut by the zero-valued contour line corresponding to $\alpha = 0$. Figure (5) which gives a clear measure of singularity along the trajectory. We can observe that the
value of $S(\phi)$ stays away from zero along the trajectory indicating that the path traced is singularity-free. Figure (7) shows the error in path-tracking. Figure (8) shows the performance of control over $\alpha$ along the trajectory. The values of $K_p$ and $K_v$ are so chosen that along with a good control we also should get feasible values of torques at all the motors. Figure (9) shows the variation of torques at three motors installed at three base points of the manipulator.

Figure 4: Initial configuration of 3-RRR: $x = 0.20$, $y = 0.21$, $\alpha = 0$

Figure 5: Variation of $S(\phi)$ along the trajectory
Figure 6: The path tracked vs. the desired path

Figure 7: Error in path-tracking
Figure 8: Variation of $\alpha$ along the trajectory

Figure 9: Variation of the joint torques
5.2 Tracing the non-singular circle with disturbances added

In the second case we trace the same circle as above with \( v = 0.2 \text{m/sec} \) but now with disturbances added into the plant. We incremented the masses and the length of the links by 15\% to introduce the errors into the model and repeated the same simulation with \( K_p, K_v \) values kept same. Figure (10) and (11) show that the error in path-tracking has increased in comparison to the first case, where there are no disturbances. Figure (12) shows the increase in torques values to compensate for these errors. We have kept the control gains same as were in the first case, so as to present a clear comparison of the two cases. However, again tuning the control gains may result in lesser path-tracking errors.

![Figure 10: The path tracked vs. the desired path with 15\% error in mass and length of all the links](image)

5.3 Tracing the non-singular circle with increased speed

In the third case we trace the same circle but with increased speed. This time we trace the circle with double the speed i.e., \( v = 0.4 \text{m/sec} \) with no disturbances. We can observe from figure (13) and (14) that the error in path-tracking is much larger in comparison to the first case, where the tracking speed was 0.2m/sec. Figure (15) shows that the torques are almost four
Figure 11: Error in path-tracking with 15% error in mass and length of all the links times of the previous value of torques at \( v = 0.2 \text{m/sec} \) (see in figure (9)).

6 Conclusion

In this paper we have presented a trajectory-tracking control scheme for 3-R\textsubscript{RR} planar parallel manipulator. It is shown that the linear control scheme works effectively for 3-R\textsubscript{RR} manipulator even with 15 percent error added in lengths and masses. However, with increase in speed of the end-effector there are large amount of deviations from the desired path and hence the same control gains do not work for the same trajectory. This requires the tuning of control gains, \( K_p, K_v \). It is proposed that the same formulation can be applied to any parallel manipulator to trace singularity-free-paths.
Figure 12: Variation of the joint torques with 15% error in mass and length of all the links

Figure 13: The path tracked vs. the desired path, with $v = 0.4\text{m/s}$
Figure 14: Error in path-tracking with \( v = 0.4\text{m/s} \)

Figure 15: Variation of the joint torques with \( v = 0.4\text{m/s} \)