ED 3160 - Fall 2011 - Daily Quizzes

September 4, 2011

1. Describe the mechanical properties desired in a cricket bat, and recommend different materials for each part.

2. Following the analysis from the previous class, derive the strength property group that one needs to look at in order to minimize mass of frame tube per unit length, if you are asked to keep the ratio R/t constant.

   - **Solution**: The maximum bending stress is

   $$\sigma_e = \frac{MR}{I} = \frac{MR}{\pi R^3 t} = \frac{M}{\pi R^2 t}$$

   In this case, the geometric condition we are given is that $\frac{R}{t}$ is to be maintained constant, equal to say, k. Thus, the above expression can be rewritten as

   $$\sigma_e = \frac{M}{\pi k^2 t^3}$$

   Thus, for a given k, the minimum thickness required to withstand the applied moment is given by

   $$t = \left[\frac{M}{\pi \sigma_e k^2}\right]^{\frac{1}{3}}$$

   Based on strength considerations, the mass per unit length is

   $$\frac{m}{L} = 2\pi R t \rho = 2\pi k^2 t \rho = 2\left[\frac{\pi}{k} M^2\right]^{\frac{1}{3}} \left(\frac{\rho}{\sigma_e^{\frac{2}{3}}}\right)$$

   Thus, the property group that must be maximized in order to minimize the mass per unit length of frame tube is

   $$P_1 = \left(\frac{\sigma_e^{\frac{2}{3}}}{\rho}\right)$$

   Using a similar argument, it can be shown that the property group that must be maximized in order to minimize the mass per unit length of frame from stiffness considerations is

   $$P_2 = \left(\frac{E^{\frac{1}{2}}}{\rho}\right)$$

3. Consider a state of pure shear stress in the 1-2 plane. If the magnitude of the shear stress is τ, what are the stress components $\sigma_{11}, \sigma_{12}, \sigma_{22}$ in a coordinate system obtained by rotating the original coordinate system 45°CCW?

   - **Solution**: The coordinate system required for the transformation is (2-D will suffice here):

   $$[Q] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad (1)$$

Then, the stress components in the new coordinate system can be expressed as

$$[\sigma'] = [Q][\sigma][Q]^T \tag{2}$$

Since the stress components in the original coordinate system are given by

$$[\sigma] = \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix}, \tag{3}$$

we get from Eqn. (2) that

$$\begin{aligned}
[\sigma] &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \tau \\ \tau & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{\tau}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{\tau}{2} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} \tau & 0 \\ 0 & -\tau \end{bmatrix}
\end{aligned} \tag{4}$$

4. Consider a bar of cross-section A extended uniaxially by a force $\mathbf{F} = F\mathbf{e}_1$. What is the expression for the stress tensor in this case? Does it satisfy the equilibrium equation?

   - **Solution:** The stress tensor components in this case are

     $$[\sigma] = \begin{bmatrix} \frac{F}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

     i.e. the stress field is homogeneous. (Actually, for this statement to be true, one needs to be 'suitably' far away from the ends where the loads are applied; we will assume so to keep matters simple). Since the stress field is spatially uniform, all its spatial derivatives are zero, which means that it satisfies the three equilibrium equations trivially in the absence of body forces and accelerations.

5. In your own words, describe the difference between the terms stiffness and strength.

6. For the motion prescribed by $x_1 = X_1 \cos\phi + X_2 \sin\phi,\ x_2 = -X_1 \sin\phi + X_2 \cos\phi,\ x_3 = X_3$, compute the infinitesimal strain components. What happens when ϕ is large? Comment on the significance of this finding.

   - **Solution**: The given motion describes a rotation of the point (X_1, X_2, X_3) about the 3-axis through an angle ϕ . The displacement is given by

     $$\begin{aligned}
     u_1 &= X_1(\cos\phi - 1) + X_2 \sin\phi \\
     u_2 &= -X_1 \sin\phi + X_2(\cos\phi - 1) \\
     u_3 &= 0
     \end{aligned}$$

     Then, the infinitesimal strain tensor components $\epsilon_{ij} = {}^1\!/\!_2(u_{i,j} + u_{j,i})$are obtained by straighforward differentiation:

     $$\begin{aligned}
     \epsilon_{11} &= \cos\phi - 1 \\
     \epsilon_{12} &= \sin\phi - \sin\phi = 0 \\
     \epsilon_{22} &= \cos\phi - 1 \\
     \epsilon_{13} &= \epsilon_{23} = \epsilon_{33} = 0
     \end{aligned}$$

Since the motion is a pure rotation, a good strain measure should yield zero strains. This is indeed true for the shear strains for all angles of rotation. However, the axial strains vanish only for small rotations. For large rotations, $\cos\phi \neq 1$, and the infinitesimal strain tensor yields spurious non-zero axial strains. This is a serious shortcoming of this strain measure.

7. What is the Poisson effect in linear elastic materials? How do we define the material parameter characterizing this effect?

8. The Young's modulus of Aluminium is 70 GPa and its shear modulus is 26.1 GPa. What is the lateral contraction of an Aluminium rod of diameter 60 mm and length 1 m if the axial elongation is 0.1 cm?

   - **Solution**: The Poisson's ratio ν can be obtained from the two elastic constants given:

   $$\nu = \frac{E}{2G} - 1 = \frac{70}{2 \times 26.1} - 1 = 0.341$$

   The axial strain is given to be $0.1\text{cm}/100\text{cm} = 0.001$. The lateral strain is then $-\nu \times 0.001 = -3.4 \times 10^{-4}$. Therefore, the lateral contraction of the rod is $3.4 \times 10^{-4} \times 60\text{mm} = 0.02\text{mm}$.

9. Consider a bar of uniform cross-section A subjected to an axial load f. Using the definition of strain energy density, compute the total elastic energy stored in the bar.

   - **Solution:** Stress at a section: $\sigma_{11} = \frac{F}{A}$
   Strain $= \epsilon_{11} = \frac{F}{AE}$
   SED: $\frac{1}{2}\sigma_{11}\epsilon_{11} = \frac{1}{2}\frac{F^2}{A^2E}$
   Since the SED is constant throughout the section, total strain energy in the bar $=$ SED $\times V = \frac{1}{2}\frac{F^2}{A^2E} \times AL = \frac{1}{2}\frac{F^2L}{AE}$
   The answer can also be arrived by considering the product of the force times the displacement:
     - Total work done $= \frac{1}{2}$Force $\times$Displacement $= \frac{1}{2}F \times \frac{FL}{AE} = \frac{1}{2}\frac{F^2L}{AE}$

   (a) Consider a bar of uniform cross-section A subjected to a torque T. Using the definition of strain energy density, compute the total elastic energy stored in the bar.

   - **Solution**: Stress at a section: $\sigma_{\rho z} = \sigma_{z\rho} = T\frac{\rho}{J}$
   Strain $= \epsilon_{\rho z} = \epsilon_{z\rho} = \frac{T\rho}{GJ}$
   S.E.D. $= \frac{1}{2}\left(\sigma_{\rho z}\epsilon_{\rho z} + \sigma_{z\rho}\epsilon_{z\rho}\right) = \frac{1}{2}\frac{T^2\rho^2}{GJ^2}$
   Total Energy $= \iint \frac{T^2\rho^2}{GJ^2}(2\pi\rho d\rho)dl = \frac{T^2}{GJ^2}JL = \frac{1}{2}\frac{T^2L}{GJ}$
   - This can also be calculated as the product of the applied torque times the total angle of twist:
   Total work done: $\frac{1}{2}T \times \theta = \frac{1}{2}T \times \frac{TL}{GJ} = \frac{1}{2}\frac{T^2L}{GJ}$

10. Derive expressions for the stress and lateral strains developed in the axially constrained bar subjected to a temperature increase, discussed in class. Assume that the axial direction is the 1-direction.

11. Compose a limerick to define plane stress or plane strain

12. Given two materials, A ($E = 310$GPa,$\sigma_y = 415$MPa)and B ($E = 207$GPa,$\sigma_y = 965$MPa),

   (a) which is stiffer and why?
      - A is stiffer because it has the higher modulus.

   (b) which is more resilient and why?
      - Modulus of resilience $U_R = \frac{\sigma_y^2}{2E}$. For A, $U_R = \frac{415e6^2}{2\times 310e9} = 2.77e5$. For B, $U_R = \frac{965e6^2}{2\times 207e9} = 2.25e6$
      - Clearly, B is more resilient.