Tutorial #3, due in class on August 26, 2011

Ground Rules

- Consultation with your team-mates, TAs or the Instructor is encouraged. However, each student is expected to write out and hand in his/her own solutions.
- Please turn in solutions to all questions today.
- **Grading**: 1 point for fully correct solution. No partial grading.

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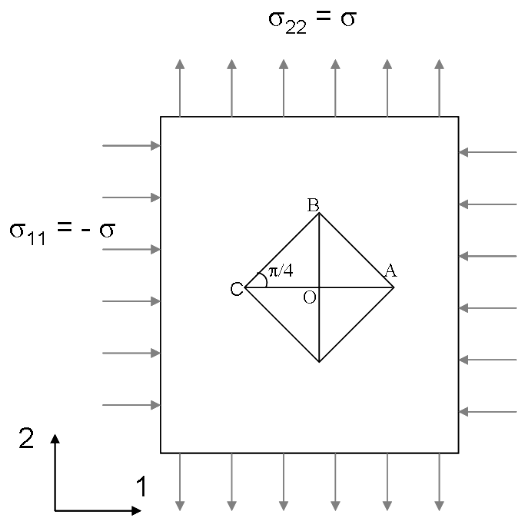
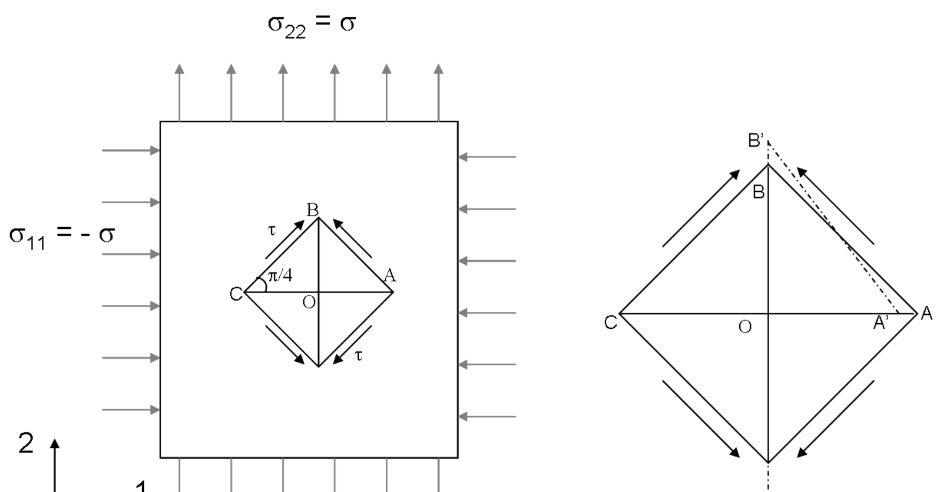
1. Consider biaxial loading of an isotropic, linear-elastic material such that $\sigma_{11} = -\sigma$, $\sigma_{22} = \sigma$.



By considering the deformation of a square of material oriented 45° to the original axes, show that one can derive the following relationship between the elastic constants:

$$G = \frac{E}{2(1+\nu)}$$

Solution: (Method 1)



Consider a case of biaxial loading such that $\sigma_{11} = -\sigma$, $\sigma_{22} = \sigma$. We have seen in class that this is a case of pure shear and that if we consider a square element oriented at 45° to the 1 or 2 axis, the normal stress components vanish and the shear stress τ is equal toσ on these planes.

If the material is isotropic and linear-elastic, we see that under this shear stress, the square element deforms in pure shear. It elongates along the 2 direction and contracts along the 1-direction. If the total shear strain is γ, then, by symmetry, the angles $\angle CBO$ and $\angle ABO$ each decrease by $\gamma/2$. Thus,

$$\tan(\frac{\pi}{4} - \frac{\gamma}{2}) = \frac{OA'}{OB'}$$

L.H.S. can be expanded using a trigonometric identity to give

$$\tan(\frac{\pi}{4} - \frac{\gamma}{2}) = \frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}}$$

R.H.S. can be computed from the geometry:

$$\frac{OA'}{OB'} = \frac{OA(1 + \epsilon_{11})}{OB(1 + \epsilon_{22})}$$

Recognizing that OA and OB are of equal length, we obtain,

$$\frac{OA'}{OB'} = \frac{1 + \epsilon_{11}}{1 + \epsilon_{22}}$$

The strains can be obtained from Hooke's law:

$$\begin{aligned} \epsilon_{11} &= -\tfrac{\sigma}{E} - \nu(\tfrac{\sigma}{E}) &= -\frac{\sigma}{E}(1+\nu) \\ \epsilon_{22} &= \tfrac{\sigma}{E} - \nu(-\tfrac{\sigma}{E}) &= \frac{\sigma}{E}(1+\nu) \end{aligned}$$

Putting everything together, we get

$$\frac{1 - \frac{\gamma}{2}}{1 + \frac{\gamma}{2}} = \frac{1 - \frac{\sigma}{E}(1+\nu)}{1 + \frac{\sigma}{E}(1+\nu)}$$

Therefore,

$$\frac{\gamma}{2} = \frac{\sigma}{E}(1+\nu)$$

However, we know that $\gamma = \tau/G$, and since $\tau = \sigma$ in this case, we get

$$\gamma = \sigma/G$$

Comparing the previous two expressions, we see that

$$G = \frac{E}{2(1+\nu)}$$

Method 2 (from Rudra): Compare the strain energy density in the original coordinate system and the one at 45° to arrive at the same answer.

2. (From Timoshenko and Goodier, Theory of Elasticity) 'An elastic layer is sandwiched between two perfectly rigid plates, to which it is bonded. The layer is compressed between the plates, the direct (i.e. normal) stress being σ_{33}. Supposing that the attachment to the plates prevents lateral strain ϵ_{11} and ϵ_{22} completely, find the *apparent Young's modulus* ($\sigma_{33}/\epsilon_{33}$) in terms of E and ν.'
   What can you say about the apparent Young's modulus for a nearly incompressible ($\nu \approx 0.5$) material?
   Solution:
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   $$\epsilon_{11} = 0 \Longrightarrow \frac{\sigma_{11}}{E} - \nu\left(\frac{\sigma_{22}}{E} + \frac{\sigma_{33}}{E}\right) = 0 \Longrightarrow \sigma_{11} = \nu(\sigma_{22} + \sigma_{33})$$
   - Similarly,
   $$\epsilon_{22} = 0 \Longrightarrow \sigma_{22} = \nu(\sigma_{11} + \sigma_{33})$$
   - Using the above two results, we get,
   $$\sigma_{11} = \sigma_{22} = \frac{\nu\sigma_{33}}{1-\nu}$$

   Then,
   $$\begin{aligned} \epsilon_{33} &= \tfrac{\sigma_{33}}{E} - \nu\left(\tfrac{\sigma_{11}}{E} + \tfrac{\sigma_{22}}{E}\right) \\ &= \tfrac{1}{E}\left(\sigma_{33} - \nu\tfrac{2\nu\sigma_{33}}{1-\nu}\right) \\ &= \tfrac{1-\nu-2\nu^2}{E(1-\nu)} \Longrightarrow E_{app} = \tfrac{\sigma_{33}}{\epsilon_{33}} = \tfrac{E(1-\nu)}{1-\nu-2\nu^2} \end{aligned}$$

3. Explain why it is much easier to push a cork into a bottle than it is to push a rubber plug of the same dimensions. Poisson's ratio for cork is 0 and that of rubber is 0.5.
   Solution: Think about the lateral expansion of the cork and rubber respectively and how that translates to a force resisting the pull. Use generalized Hooke's law to express your answer mathematically.