Tutorial #6, due in class on Oct 15, 2011

1. **Stress Concentration around a hole (from Z. Suo, Harvard):** Stress concentration at geometric discontinuities is the most important practical result in elasticity theory. For example, for a small circular hole in a large plate under uniaxial stress σ , the elasticity solution gives the hoop stress around the hole:

$$\sigma_{\theta\theta} = \sigma(1 - 2\cos 2\theta).$$

Here the polar angle ϑ is measured from the loading direction. The problem is solved in all textbooks of linear elasticity.

   (a) Under uniaxial tension, indicate the highest tensile stress around the hole.
   - **Solution:** The highest tensile stress around the hole is obtained when $\cos 2\theta = -1$, in which case the value is 3σ. Thus, we get a maximum stress concentration around the hole of 3.

   (b) Under uniaxial compression, indicate the highest tensile stress around the hole.
   - **Solution:** Since the applied stress is compressive (here), the highest tensile (positive) stress around the hole is obtained when $\cos 2\theta = +1$, in which case the value is σ.

   (c) Use the above solution and linear superposition to calculate the stress concentration at the hole when the plate is under a shear stress.
   - **Solution:** A pure shear stress state of, say, σ is equivalent to the simultaneous application of a tensile stress of σ applied at 45° and a compressive stress of magnitude σ applied along$-45°$ to the original 1-axis. This means that the stress concentration around the hole (defined as the highest tensile stress divided by the applied stress magnitude) can be obtained by adding (superposing) the solutions from each of these stresses, which have been calculated in parts (a) and (b) above. Using this approach, the stress concentration is found to be equal to 4.

2. **Griffith's Model for Experimental Strength (from Z. Suo, Harvard):** Griffith found that the strength of a material relates to the surface energy, elastic modulus, and flaw size, namely,

$$\sigma_F = \sqrt{\frac{2\gamma E}{\pi a}}$$

Consequently, strength is not a material property, but depends on the flaw size. Consider the glass used by Griffith, with surface energy γ = 1.75 J/m2 and Young's modulus E = 62 GPa.

   (a) Calculate the tensile strength of a large glass panel with defect size $a = 1\mu m$.
   - **Solution:** The strength is found by applying the formula above to be equal to 263 MPa.

   (b) The same glass panel now has a small circular hole of radius a = 1 mm, assuming that the defect size remains to be $a = 1\mu m$. How much load can the panel carry?
   - **Solution:** The hole acts to amplify the applied stress by a factor of 3. Therefore, the tensile strength of 263 MPa will be reached when the applied load is only third of this value, or 88 MPa.

   (c) What if the circular hole is replaced by a crack-like notch with half-width a = 1 mm?
   - **Solution:** The strength, by the above formula, drops to 8.3 MPa.

3. A sheet of glass measuring 2 m by 200 mm by 2 mm contains a central slit parallel to the 200 mm side. The sheet is restrained at one end and loaded in tension with a mass of 500 kg. What is the maximum allowable length of slit before fracture occurs? Assume the following material property values: E = 60 GPa, surface energy is 0.5 $J\,\mathrm{m}^2$, Poissons ratio = 0.25 and the fracture stress of sound glass is 170 MPa.
   - **Solution:** The slit is much smaller than the sheet, so the formula of problem 2 can be used. Therefore the length of the slit, $2a$, is

$$2a = 2\frac{2\gamma E}{\pi\sigma_F^2}$$

     – Recognizing that $\sigma_F = 500 \times 9.81/(0.2 \times 0.002) = 12.26$ MPa, we get $2a = 0.254$mm. The fracture strength of sound (uncracked) glass is given in order to serve as a reference value to compare the strength of the cracked glass sheet to. Notice that because of the crack, the strength is reduced by more than a factor of 10.

4. Griffith listed in his paper, for the glass he used, Young's modulus 62 GPa, tensile strength 170 MPa, and surface tension 1.75 J/m2. Estimate the flaw size in the glass. Compare it with the typical molecular size and sample size.

- **Solution:** We use the equation of problem 3 again to obtain

$$2a = \frac{4 \times 1.75 \times 62 \times 10^9}{3.14 \times (170 \times 10^{6)^2}} = 4.78 \mu m$$

- Typical molecular size is of the order of Angstroms ($10^{-10} m$) and sample size is of the order of mm to m. Therefore, the flaw size is much larger than molecular dimensions, but at least 3 orders of magnitude smaller than typical sample dimensions.