Tutorial #7, due in class on Oct. 25, 2010

1. Using the Mode I crack-tip intensity fields discussed in class and the von Mises yield criterion,

    (a) Derive expressions for the plane-stress and plane-strain plastic zone radii as functions of θ.

    - **Solution:** The principal stresses under mode I loading can be found from the stress solution given in class:

$$\begin{aligned}
\sigma_1 &= \frac{K_I}{(2\pi r)^{\frac{1}{2}}}(1+\sin\frac{\theta}{2})\cos\frac{\theta}{2} \\
\sigma_2 &= \frac{K_I}{(2\pi r)^{\frac{1}{2}}}(1-\sin\frac{\theta}{2})\cos\frac{\theta}{2} \\
\sigma_3 &= \begin{cases} 0 & \text{, plane stress} \\ \frac{2\nu K_I}{(2\pi r)^{\frac{1}{2}}}\cos\frac{\theta}{2}, & \text{plane strain} \end{cases}
\end{aligned}$$

    Using these in the von Mises yield criterion with yield stress σ_y, we get

$$r_y(\theta) = \begin{cases} \frac{K_I^2}{4\pi\sigma_y^2}[1+\cos\theta+\frac{3}{2}\sin^2\theta] & \text{, plane stress} \\ \frac{K_I^2}{4\pi\sigma_y^2}[(1-2\nu)^2(1+\cos\theta)+\frac{3}{2}\sin^2\theta] & \text{, plane strain} \end{cases}$$

    (b) What is the value of θ that maximizes the radius for each case?

    - **Solution:** The maximum radius under plane stress conditions is at $\theta = 70.5°$. Assuming $\nu = 0.3$, maximum radius under plane strain is at $\theta = 86.9°$.

2. A sharp penny-shaped crack with a diameter of 2.5 cm is completely embedded in a solid. Catastrophic fracture occurs when a stress fo 700 MPa is applied. What is the fracture toughness of the material? (Assume that this event constitutes a valid fracture toughness test.)

    - **Solution:** The stress intensity factor (from 'The Stress Analysis of Cracks Handbook') for a mode I penny-shaped crack is $K_I = \frac{2}{\pi}\sigma\sqrt{\pi a}$, where σ is the eapplied stress and a is the crack radius. Using the given values, we obtain

$$\begin{aligned}
K_{IC} &= 2 \times (700\text{MPa})\sqrt{\frac{1.25\times10^{-2}\text{m}}{3.14}} \\
&= 88.3\text{MPa}\sqrt{\text{m}}
\end{aligned}$$

3. A cylindrical pressure vessel, with a diameter of 6.1 m and a wall thickness of 25.4 mm, underwent catastrophic fracture when the internal pressure reached 17.5 MPa.

    The steel of the pressure vessel had E = 210 GPa, a yield strength of 2450 MPa a value of $G_c = 131 \text{kJ/m}^2$.

    (a) Show that failure would not have been expected if the von Mises yield criterion had served for design purposes.

    - **Solution:** For a pressure vessel, the three principal stresses (p is the internal presure, r is the mean radius and t is the thickness) are

$$\begin{aligned}
\text{Hoop Stress}: \quad \sigma_1 &= \frac{pr}{t} \\
\text{Axial Stress}: \quad \sigma_2 &= \frac{pr}{2t} \\
\text{Radial Stress}: \quad \sigma_3 &= \begin{cases} 0 & \text{at r} = \text{r}_\text{o} \\ -p & \text{at } r = r_i \end{cases}
\end{aligned}$$

    Thus, $\sigma_2 = 17.5\times3.05/(2\times0.0254) = 1.05\text{GPa}$, $\sigma_1 = 2.1\text{GPa}$ and at the interior wall (where the distortion energy will be the highest along the radial direction), $\sigma_3 = -17.5\text{MPa}$. Using the formula for J_2, we get $J_2 = 1.12\text{GPa}^2$. Thus, the von Mises equivalent stress ($=\sqrt{3J_2}$) is 1.83 GPa, which is less than the yield stress of 2.45 GPa. Therefore, the material should not fail according to the von Mises criterion.

    (a) Based on Griffith's analysis determine the size of crack that might have caused this failure, stating assumptions that you have made.

- **Solution:** A crack oriented along the axial direction (i.e. perpendicular to the hoop stress, which is the largest principal stress) would represent the worst case with respect to fracture. We can use Griffith analysis to calculate the size of the crack:

$$
\begin{aligned}
2a &= 2\frac{G_c E}{\pi \sigma_F^2} \\
&= \frac{2 \times 131000 \times 210 \times 10^9}{3.14 \times (2.1 \times 10^{9)^2}} \\
&= 4\text{mm}
\end{aligned}
$$